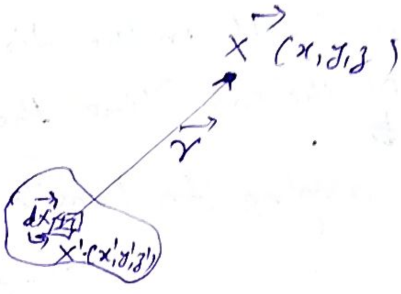
Curl of $\vec{B}$:

Magnetic field at point $\vec{x} = (x, y, z)$ is given by

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}') \times \hat{r}}{r^2} \, d\vec{x}' \quad \text{--- (1)}$$



$$\vec{x} = (x, y, z)$$
$$\vec{x}' = (x', y', z')$$
$$d\vec{x}' = dx' dy' dz'$$
$$\vec{r} = \vec{x} - \vec{x}'$$
$$r = |\vec{r}| = |\vec{x} - \vec{x}'|$$

Taking curl on both sides of eq^n (1)

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left[\vec{J} \times \frac{\hat{r}}{r^2} \right] d\vec{x}' \quad \text{--- (2)}$$

Taking

$$\nabla \times \left[\vec{J} \times \frac{\vec{r}}{r^3} \right]$$

In above expression we use the identity

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B} + \vec{A} (\nabla \cdot \vec{B}) - \vec{B} (\nabla \cdot \vec{A})$$

$$\nabla \times \left[\vec{J} \times \frac{\vec{r}}{r^3} \right] = \left(\frac{\vec{r}}{r^3} \cdot \nabla \right) \vec{J} - (\vec{J} \cdot \nabla) \frac{\vec{r}}{r^3} + \vec{J} \left(\nabla \cdot \frac{\vec{r}}{r^3} \right) - \frac{\vec{r}}{r^3} (\nabla \cdot \vec{J})$$

Since $\vec{J}$ is function of $\vec{x}' = (x', y', z')$, therefore the first term and the last term of above equation is zero and we obtain

$$\nabla \times \left[\vec{J} \times \frac{\vec{r}}{r^3} \right] = -(\vec{J} \cdot \nabla) \frac{\vec{r}}{r^3} + \vec{J} \left(\nabla \cdot \frac{\vec{r}}{r^3} \right) \quad \text{--- (3)}$$

Next, taking first term of eq^n (3), we can write

$$-(\vec{J}\cdot\nabla)\frac{\vec{r}}{r^3} = (\vec{J}\cdot\nabla')\frac{\vec{r}}{r^3} \quad \text{—— (4)}$$

In above expression we have taken $\nabla \equiv -\nabla'$

because grad acts on $\frac{\hat{\vec{r}}}{r^3}$, which depends on difference

of $\vec{x}$ and $\vec{x}'$,

$$\frac{\partial}{\partial x} f(x-x') = -\frac{\partial}{\partial x'} f(x-x')$$

Now the x component of eq^n (4) can be written as

$$(\vec{J}\cdot\vec{\nabla}')\left[\frac{x-x'}{r^3}\right] = \nabla'\cdot\left[\frac{x-x'}{r^3}\vec{J}\right] - \left(\frac{x-x'}{r^3}\right)(\nabla'\cdot\vec{J}) \quad (5)$$

where we have used the following identity

$$\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f)$$

$$\nabla\cdot\left[\frac{x-x'}{r^3}\vec{J}\right] = \frac{(x-x')}{r^3}(\nabla\cdot\vec{J}) + (\vec{J}\cdot\nabla)\left(\frac{x-x'}{r^3}\right)$$

or $$(\vec{J}\cdot\nabla)\left(\frac{x-x'}{r^3}\right) = \nabla\cdot\left[\frac{x-x'}{r^3}\vec{J}\right] - \left(\frac{x-x'}{r^3}\right)(\nabla\cdot\vec{J})$$

From the continuity equation $\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0$ we have seen that for steady currents $\nabla\cdot\vec{J} = 0$ since $\frac{\partial\rho}{\partial t} = 0$, therefore, second term of equation (5) contributes to zero.

$$(\vec{J}\cdot\nabla')\left[\frac{x-x'}{r^3}\right] = \nabla'\cdot\left[\frac{x-x'}{r^3}\vec{J}\right]$$

Volume integral of above eq^n for eq^n (2) {

$$\int_V \nabla'\cdot\left[\frac{x-x'}{r^3}\vec{J}\right] d\vec{x}' = \oint_S \left(\frac{x-x'}{r^3}\right)\vec{J}\cdot d\vec{s}' \quad \text{—— (6)}$$

where we have used the Gauss's theorem to reduce volume integral to the surface integral (surface bounds the volume)

$$\boxed{\int_V (\nabla\cdot\vec{A})\,d\vec{x} = \oint_S \vec{A}\cdot d\vec{s}}$$

H.W. Check the Gauss's theorem using the function $\vec{A} = y^2\hat{i} + (2xy+z^2)\hat{j} + (2yz)\hat{k}$ and the unit cube situated at the origin.

~~Now eqn~~ Since the current is zero on the boundary therefore

$$\int_V \nabla'\cdot\left[\frac{x-x'}{r^3}\vec{J}\right]d\vec{x}' = 0$$

And from eqn (5)

$$\int_V (\vec{J}\cdot\nabla')\left(\frac{x-x'}{r^3}\right)d\vec{x}' = 0 \qquad (7)$$

{first and second term of eqn (5) goes to zero

Therefore, using eqn (7), (3) and (2), we obtain

$$\nabla\times\vec{B} = \frac{\mu_0}{4\pi}\int \nabla\times\left[\vec{J}\times\frac{\vec{r}}{r^3}\right]d\vec{x}'$$

$$= \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\left(\nabla\cdot\frac{\vec{r}}{r^3}\right)d\vec{x}'$$

using the identity $\nabla\cdot\left(\frac{\vec{r}}{r^3}\right) = 4\pi\,\delta^3(\vec{x}-\vec{x}')$ {H.W., Prove the identity}

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}')\, 4\pi \delta^3(\vec{x}-\vec{x}')\, d\vec{x}'$$

$$= \frac{\mu_0}{4\pi} 4\pi \cdot \vec{J}(\vec{x})$$

or $\boxed{\nabla \times \vec{B} = \mu_0 \vec{J}}$ Ampère's law in differential form